JHEP HYPER VERSION

RECEIVED: *November 09, 2000,* ACCEPTED: *December 28, 2000*

A note on the equivalence of Vafa's and Douglas's picture of discrete torsion

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ABSTRACT: For a general non-abelian group action and an arbitrary genus world-sheet we show that Vafa's old definition of discrete torsion coincides with Douglas's D-brane definition of discrete torsion associated to projective representations.

KEYWORDS: Superstrings and Heterotic Strings, Superstring Vacua, Discrete and Finite Symmetries.

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1. Vafa's picture

Discrete torsion was introduced many years ago by Vafa in [1]. More recently Douglas [2] introduced an alternative picture in terms of D-branes. The purpose of this note is to show the equivalence of these pictures. Since both of these papers were very brief as regards to the general case we will review both constructions here.

In general a worldsheet Σ of genus g and no boundary will be associated to a given phase dictated by the B-field and the homology class of the worldsheet in the target space. Vafa wanted to generalize this notion of a B-field to the case of orbifolds with fixed points.

In order to localize the picture let us consider a discrete group Γ acting on $\mathbb{C}^n$ where the origin is fixed by all of Γ. The worldsheet Σ on this orbifold may be pictured as a disk in the covering $\mathbb{C}^n$ where the edges of the disk are identified by elements of the action of Γ. That is, the usual homology 1-cycles of the worldsheet Σ are twisted by elements of Γ.

The twisting of these cycles in Σ may be viewed as a Γ-bundle on Σ. The holonomy of this bundle is given precisely by the twist associated to a given loop. Since Γ is a discrete group, this bundle is flat. Such a bundle is classified by the homotopy class of a mapping $\phi : \Sigma \to B\Gamma$, where $B\Gamma$ is the classifying space for the group Γ. That is, $\pi_1(B\Gamma) = \Gamma$ and $\pi_n(B\Gamma) = 0$ for $n > 1$.

In this way, $B\Gamma$ in the case of an orbifold plays a rather similar role to the target space of the usual string theory. Vafa's idea was to define the discrete torsion version of the B-field as $H^2(B\Gamma, \mathrm{U}(1))$ because of this analogy. The connection between the B-field and group cohomology has been clarified recently by the work of Sharpe [3].

Now group (co)homology, which is defined as the (co)homology of $B\Gamma$, can be described algebraically directly in terms of Γ as we now show. In order to complete the description of discrete torsion we need to describe the above picture in terms of this more intrinsic definition.

Let $\mathbb{Z}\Gamma$ be the *group ring* of Γ. That is, any element of $\mathbb{Z}\Gamma$ may be written uniquely as $\sum_{g\in\Gamma} a_g g$ for $a_g \in \mathbb{Z}$. Now let

$$\cdots \longrightarrow F_n \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow \mathbb{Z} \longrightarrow 0\,, \tag{1.1}$$

be a free resolution of $\mathbb{Z}$ as a $\mathbb{Z}\Gamma$-module. This is an exact sequence of $\mathbb{Z}\Gamma$-modules where F_n is a free $\mathbb{Z}\Gamma$-module for any n. The Γ-action on $\mathbb{Z}$ is taken to be trivial.

Now define $(F_n)_\Gamma$ as the Γ-coinvariant projection of F_n. That is, we divide F_n by the equivalence $g \cong 1$ for any $g \in \Gamma$. Since F_n is a free $\mathbb{Z}\Gamma$-module, $(F_n)_\Gamma$ will be a free $\mathbb{Z}$-module, i.e. a free abelian group. The homology of the induced complex

$$\cdots \longrightarrow (F_n)_\Gamma \longrightarrow \cdots \longrightarrow (F_1)_\Gamma \longrightarrow (F_0)_\Gamma \longrightarrow 0 \tag{1.2}$$

is then equal to $H_n(\Gamma)$, the homology of Γ. See [4, section II.4], for example, for a proof that this equals $H_n(B\Gamma)$.

One way to explicitly compute $H_n(\Gamma)$ is via the "bar resolution" as follows.[1] Let F_n be generated by $(n+1)$-tuples of the form $(g_0, g_1, \ldots, g_n)$ where the Γ-action is defined as

$$g:\ (g_0, g_1, \ldots, g_n) \longmapsto (gg_0, gg_1, \ldots, gg_n)\,. \tag{1.3}$$

The boundary map $\partial : F_n \to F_{n-1}$ is defined as

$$\partial:\ (g_0, g_1, \ldots, g_n) \longmapsto \sum_{i=0}^{n} (-1)^i (g_0, g_1, \ldots, \widehat{g_i}, \ldots, g_n)\,, \tag{1.4}$$

where the hat indicates omission as usual. Because of the Γ-action, we may use $(1, g_1, \ldots, g_n)$ as a basis for F_n or $(F_n)_\Gamma$. The "bar" notation is to write

$$[g_1|g_2|\cdots|g_n] = (1, g_1, g_1g_2, \ldots, g_1g_2\cdots g_n)\,. \tag{1.5}$$

We will generally consider the homology of Γ in terms of the generators $[g_1|g_2|\cdots|g_n]$. One may show that a generator may be considered trivial if any of the entries $g_1, g_2, \ldots$ are equal to 1. It is also useful to note the explicit form of the boundary map in (1.2) of a 3-chain in the bar notation:

$$\partial[a|b|c] = [b|c] - [ab|c] + [a|bc] - [a|b]\,. \tag{1.6}$$

So how is the homology class of Σ in $B\Gamma$ described in terms of these bar chains? The free resolution (1.1) is actually an augmented simplicial chain complex, Δ, as follows. Let the elements of Γ be viewed as the vertices of Δ. Now let $(g_0, g_1, \ldots, g_n)$ be the n-dimensional simplex in Δ with the corresponding vertices. Thus we have exactly one simplex in Δ for every possible sequence of group elements. The group Γ acts on Δ to give the complex (1.2) as a quotient. As far as homology is concerned, the abstract simplicial complex Δ/Γ is a perfectly good representative for $B\Gamma$.

[1] In practice this method is usually very inefficient!

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We may now take a simplicial decomposition of Σ and explicitly map it into this simplicial picture of $B\Gamma$. This will relate the homology of Σ, and in particular its fundamental class, to the homology of Γ. As a simple example let us consider a torus as shown in figure 1. This torus consists of one cycle twisted by the action of $a \in \Gamma$ and another cycle twisted by the action of $b \in \Gamma$. The topology of the torus dictates that $ab = ba$. The figure shows the labels of the vertices in terms of the group action and it also shows a simple simplicial decomposition. We are free to label one vertex as "1". In terms of the image of this torus in our peculiar simplicial model of $B\Gamma$ we see that U corresponds to a simplex $(1, a, ab)$ and L corresponds to a simplex $(1, b, ab)$. As usual in simplicial homology one needs to be careful about relative signs. The fundamental class of Σ is given by $U - L$ so that the diagonal line in figure 1 cancels for the boundary of $U - L$. This gives

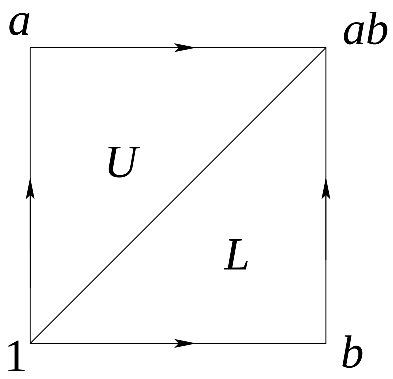

Figure 1: A simplicial decomposition of a torus.

$$\begin{aligned}
[\Sigma] &= (1, a, ab) - (1, b, ab) \\
&= [a|b] - [b|a]\,,
\end{aligned} \tag{1.7}$$

in terms of the bar chains.

Applying $\mathrm{Hom}(-, \mathrm{U}(1))$ to this bar resolution we may go over to the group cohomology picture. A group n-cochain can now be viewed as a map from $(\Gamma)^n$ to $\mathrm{U}(1)$. For example, if $\alpha \in H^2(\Gamma, \mathrm{U}(1))$ then $\alpha(a, b) \in \mathrm{U}(1)$ for $a, b \in \Gamma$. From (1.6) we see that a 2-cochain is closed if

$$\begin{aligned}
(\delta\alpha)(a, b, c) &= \frac{\alpha(a, bc)\alpha(b, c)}{\alpha(a, b)\alpha(ab, c)} \\
&= 1\,.
\end{aligned} \tag{1.8}$$

If discrete torsion is written in terms of the α cocycles then (1.7) is translated in Vafa's language into the statement that the phase associated to a genus one Riemann surface is given by

$$\xi_1 = \frac{\alpha(a, b)}{\alpha(b, a)}\,. \tag{1.9}$$

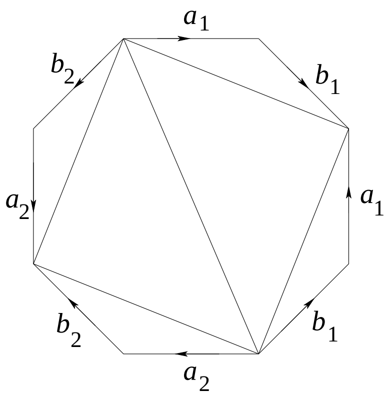

Figure 2: A simplicial decomposition of a genus 2 surface.

This is the result quoted in [1]. This method may be applied equally well to higher genus worldsheets. The genus two case is shown in figure 2 together with a specific simplicial decomposition. It follows that

$$[\Sigma] = [a_1|b_1] - [\gamma|b_1 a_1] - [b_1|a_1] + [a_2|b_2] + [\gamma|a_2 b_2] - [b_2|a_2]\,, \tag{1.10}$$

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where $\gamma = a_1 b_1 a_1^{-1} b_1^{-1}$, which by the topology of a genus two surface must equal $b_2 a_2 b_2^{-1} a_2^{-1}$. Note that if $\gamma = 1$, which would happen if Γ were abelian for example, then the genus two homology class simply breaks into $([a_1|b_1] - [b_1|a_1]) + ([a_2|b_2] - [b_2|a_2])$ which is the sum of two genus one classes. This does *not* happen in general if $\gamma \neq 1$. One may view this statement in terms of a genus two surface degenerating into two genus one surfaces touching at a point. Such a degeneration is topologically obstructed if $\gamma \neq 1$.

There are many ways of performing a simplicial decomposition of Σ in general. One may show that changing the decomposition will simply change $[\Sigma]$ in the computations above by the boundary of a group 3-chain and hence has no effect.

2. Douglas's picture

In [2] Douglas introduced another definition of discrete torsion associated to projective representations which we now review. See [5, 6] for further discussion of this construction.

Consider the following central extension of Γ:

$$1 \longrightarrow \mathrm{U}(1) \xrightarrow{i} \hat{\Gamma} \overset{\overset{s}{\curvearrowleft}}{\xrightarrow{j}} \Gamma \longrightarrow 1\,, \tag{2.1}$$

where s is a set-theoretic map such that js is the identity on Γ. Such extensions are classified by $H^2(\Gamma, \mathrm{U}(1))$. The map s defines a projective representation of Γ. Given $\alpha \in H^2(\Gamma, \mathrm{U}(1))$ written in terms of the bar resolution of the previous section one may show that [7]

$$s(a)s(b) = \alpha(a,b)s(ab)\,. \tag{2.2}$$

Douglas considered s as a lift of an orbifold action to the Chan-Paton factors on the end of an open string. He then considered a Riemann surface with a disk removed. Such a surface can represent either a multi-loop open string diagram or a multi-loop tadpole-like diagram for a closed string.

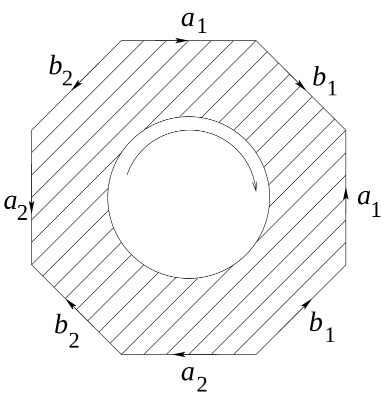

Figure 3: A genus 2 surface with a hole cut out.

Consider the genus two case shown in figure 3. When computing the amplitude of this diagram in the open string context one must include a trace of the group actions on the Chan-Paton elements along the boundary. As is clear from the figure, this boundary may be contracted to the outer polygon which is a sequence of 1-cycles which are twisted by elements of Γ. Let ξ_2 be this weighting of the amplitude. We then have

$$\xi_2 = s(a_1)s(b_1)s(a_1)^{-1}s(b_1)^{-1}s(a_2)s(b_2)s(a_2)^{-1}s(b_2)^{-1}\,, \tag{2.3}$$

where $a_i, b_i \in \Gamma$ represent the associated twists. For a general genus g surface we clearly have

$$\xi_g = \prod_{i=1}^{g} s(a_i)s(b_i)s(a_i)^{-1}s(b_i)^{-1}\,. \tag{2.4}$$

Let $\gamma_i = a_i b_i a_i^{-1} b_i^{-1}$. Then the topology of the surface dictates that $\prod_{i=1}^{g}\gamma_i = 1$.

Note that $s(1) = 1$ and thus $s(x)s(x^{-1}) = \alpha(x, x^{-1})$. It follows that $\alpha(x, x^{-1}) = \alpha(x^{-1}, x)$. Repeated use of (2.2) and (1.8) then gives[2]

$$\begin{aligned}
\xi_g &= \prod_{i=1}^{g} \frac{s(a_i)s(b_i)s(a_i^{-1})s(b_i^{-1})}{\alpha(a_i, a_i^{-1})\alpha(b_i, b_i^{-1})} \\
&= \frac{\alpha(a_1,b_1)\alpha(a_1b_1,a_1^{-1})\alpha(a_1b_1a_1^{-1},b_1^{-1})\cdots\alpha((\prod_{i=1}^{g-1}\gamma_i)a_gb_ga_g^{-1},b_g^{-1})}{\prod_{i=1}^{g}\alpha(a_i,a_i^{-1})\alpha(b_i,b_i^{-1})} \\
&= \frac{\alpha(a_1,b_1)\alpha(\gamma_1b_1a_1,a_1^{-1})\alpha(\gamma_1b_1,b_1^{-1})\cdots\alpha(b_g,b_g^{-1})}{\prod_{i=1}^{g}\alpha(a_i,a_i^{-1})\alpha(b_i,b_i^{-1})} \\
&= \frac{\alpha(a_1,b_1)}{\alpha(\gamma_1b_1,a_1)\alpha(\gamma_1,b_1)}\cdot\prod_{i=2}^{g-1}\frac{\alpha(\zeta_i,a_i)\alpha(\zeta_ia_i,b_i)}{\alpha(\zeta_i\gamma_ib_i,a_i)\alpha(\zeta_i\gamma_i,b_i)}\cdot\frac{\alpha(\zeta_g,a_g)\alpha(\zeta_ga_g,b_g)}{\alpha(b_g,a_g)}\,,
\end{aligned} \tag{2.5}$$

where $\zeta_i = \gamma_1\gamma_2\ldots\gamma_{i-1}$.

This formula was also derived in [8]. Now it is not hard to see that the final form of (2.5) corresponds to a simplicial decomposition of a genus g surface into $4g-2$ triangles in the language of section 1. Each α factor represents one triangle oriented in just the right way to build up the complete surface. We show the four triangles for a generic pair a_i, b_i in figure 4.

This factor ξ_g is exactly the same factor as we obtain by Vafa's method of the previous section applied to this simplicial decomposition. Thus we see that Vafa's and Douglas's definition of discrete torsion agree in general.

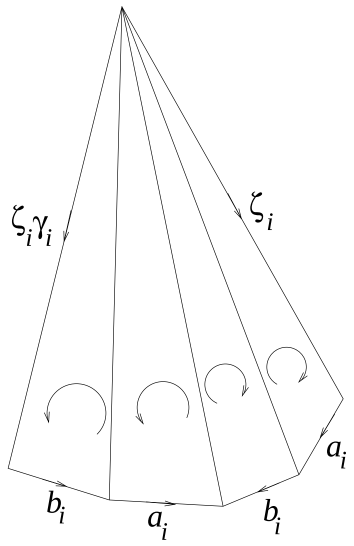

Figure 4: Part of the simplicial decomposition of the arbitrary genus case.

Acknowledgments

It is a pleasure to thank S. Katz, R. Plesser and E. Sharpe for useful conversations. The author is supported in part by NSF grant DMS-0074072 and a research fellowship from the Alfred P. Sloan Foundation.

[2] We assume $g > 1$. The case $g = 1$ is left as an easy exercise for the reader.

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